Alpha Individual

1. If α and β are two roots of the equation $5x^2 - 2x - 1 = 0$, what is $\frac{1}{\alpha} + \frac{1}{\beta}$?

   A) $\frac{1}{2}$ B) $-\frac{1}{2}$ C) -2 D) 2 E) NOTA

   Solution: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{\frac{2}{5}}{-\frac{1}{5}} = -2$

   Answer: C)

2. If $\tan x + \tan y = 6$ and $\cot x + \cot y = 3$, what is $\tan(x+y)$?

   A) -6 B) -5 C) 5 D) 6 E) NOTA

   Solution: $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{6}{1-2} = -6$

   Answer: A)

3. Suppose that $f(4-x) = 2x^2 - x - 7$ and $f(x) = px^2 + qx + r$. What is $p + q + r$?

   A) -6 B) 8 C) 14 D) -4 E) NOTA

   Solution: $f(1) = p + q + r = f(4-3) = 2(3)^2 - 3 - 7 = 8$

   Answer: B)

4. If a_n is a geometric sequence with $a_1 = 2$ and $a_5 = 18$, find the sum $a_1 + a_3 + a_5 + a_7$.

   A) 80 B) 72 C) 36 D) 27 E) NOTA

   Solution: $a_5 = 2r^4 = 18$, so $r^2 = 3$.

   $a_1 + a_3 + a_5 + a_7 = a_1(1 + r^2 + r^4 + r^6) = 80$

   Answer: A)

5. Simplify the product: $\tan 10° \tan 20° \tan 30° \cdots \tan 80°$

   A) $\frac{1}{2}$ B) 1 C) $\frac{1}{3}$ D) 3 E) NOTA

   Solution: $\tan 10° \tan 20° \tan 30° \cdots \tan 80° = \frac{\sin 10°}{\cos 10°}\frac{\sin 20°}{\cos 20°}\cdots\frac{\sin 80°}{\cos 80°} = \frac{\sin 10°}{\sin 80°}\frac{\sin 20°}{\sin 70°}\cdots\frac{\sin 80°}{\sin 10°} = 1$

   Answer: B)

6. Let $f(x) = \log_2(x + \sqrt{x^2+1})$. If $f(a) = b$, what is $f(-a)$?

A) a B) $a+b$ C) b D) $-b$ E) NOTA

Solution: $f(-a) = \log_2\left(-a+\sqrt{a^2+1}\right)$

$$= \log_2 \frac{(a^2+1)-a^2}{a+\sqrt{a^2+1}} = \log_2\left(\frac{1}{a+\sqrt{a^2+1}}\right) = \log_2\left(a+\sqrt{a^2+1}\right)^{-1} = -b$$

Answer: D)

7. Find the value of $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$.

A) 2 B) $\frac{1+\sqrt{5}}{2}$ C) $\frac{1+\sqrt{5}}{4}$ D) $\sqrt{2}$ E) NOTA

Solution: Let $A = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$. Then $A = \sqrt{1+A}$, so A is the positive root of

$A^2 - A - 1 = 0$.

Answer: B)

8. Two numbers, x and y, are selected at random from the interval $[0,2]$. What is the probability that $y \le x+1$?

A) $\frac{7}{8}$ B) $\frac{3}{4}$ C) $\frac{2}{3}$ D) $\frac{1}{2}$ E) NOTA

Solution: The area of the triangle represented by $y \ge x+1$ in the square by $0 \le x \le 2$ and

$0 \le y \le 2$ is $\frac{1}{2}$.

Answer: A)

9. How many real solutions are there to the equation $|2x-3| + |5-2x| = 2$?

A) 1 B) 2 C) 3 D) 4 E) NOTA

Solution: All real values from $\frac{3}{2} \le x \le \frac{5}{2}$ satisfy the equation.

Answer: E)

10. If $\csc x - \cot x = 7$, what is $\csc x + \cot x$?

A) 1 B) 3 C) $\frac{1}{3}$ D) $\frac{1}{7}$ E) NOTA

Solution: Since $\csc x - \cot x = \frac{1-\cos x}{\sin x} = 7$,

$$\csc x + \cot x = \frac{1+\cos x}{\sin x} = \frac{1-\cos^2 x}{\sin x\,(1-\cos x)} = \frac{\sin x}{1-\cos x} = \frac{1}{7}$$

Answer: D)

11. How many prime factors are there in 999,999?
A) 3 B) 4 C) 5 D) 6 E) NOTA

Solution: $999{,}999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$
Answer: C)

12. If $2+3i$ and $1+4i$ are two roots of the equation $x^4+ax^3+bx^2+cx+d=0$ where a,b,c,d are integers, what is the value of $a+b+c+d$?
A) 145 B) 159 C) 221 D) 230 E) NOTA

Solution: The other roots are $2-3i$ and $1-4i$, so $f(x)=x^4+ax^3+bx^2+cx+d=(x^2-4x+13)(x^2-2x+17)$. Then $a+b+c+d=f(1)-1=159$.

Answer: B)

13. Consider a rational function $f(x)=\frac{ax+b}{cx+d}$ where $c>0$. If f has its inverse function $f^{-1}(x)=\frac{x+4}{2x+1}$. What is $a+b+c+d$?
A) 5 B) -5 C) 4 D) -4 E) NOTA

Solution: The inverse function of $f^{-1}(x)$ is $f(x)$, so $f(x)=(f^{-1})^{-1}(x)=\frac{-x+4}{2x-1}$.
Answer: C)

14. How many solutions to the equation $\sin^4 x+\cos^4 x=1$ are there in the interval $[0,2\pi)$?
A) 4 B) 5 C) 6 D) 7 E) NOTA

Solution: Let $t=\sin x$, then $t^4+(1-t^2)^2=1$, and hence $t=0$, or $t=\pm 1$. There are four roots of the equation over the given interval; $x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$
Answer: A)

15. Suppose that two positive numbers x and y satisfy $\log_y x+\log_x y=\frac{10}{3}$ and $xy=81$. What is the value of $x+y$?
A) 27 B) 30 C) 36 D) 49 E) NOTA

Solution: Note that $\log_y x+\log_x y=\log_y x+\frac{1}{\log_y x}=\frac{10}{3}$. By multiplying by $3\log_y x$, we obtain $3\,(\log_y x)^2-10(\log_y x)+3=(3\log_y x-1)(\log_y x-3)=0$. Thus, $\log_y x=\frac{1}{3}$ or $\log_y x=3$, which yields $x^3=y$ or $x=y^3$. Then two pairs of solutions are $x=3, y=27$ or $x=27, y=3$. Thus in either case $x+y=30$.

Answer: B)

16. How many 4-digit numbers are there whose digit sum equals 10?
A) 200 B) 219 C) 220 D) 286 E) NOTA

Solution: Let $abcd$ represent a 4-digit number. Then $a+b+c+d=10$ for $1 \leq a \leq 9$ and $0 \leq b,c,d \leq 9$. The equation is equivalent to $a+b+c+d=9$ for $0 \leq a \leq 8$ and $0 \leq b,c,d \leq 9$. By Balls and Urns formula, there are $\binom{9+4-1}{4-1} - 1 = 219$ such 4-digit numbers.

Answer: B)

17. Suppose that $f(x)$ is a monic polynomial of degree 3 such that $f(1)=1, f(2)=4, f(3)=9$. Find the value of $f(4)$.
A) 16 B) 20 C) 22 D) 29 E) NOTA

Solution: Define a new function $g(x)$ by $g(x)=f(x)-x^2$.
Then $g(1)=f(1)-1^1=0, g(2)=f(2)-2^2=0$ and $g(3)=f(3)-3^2=0$.
Therefore $g(x)$ is a monic cubic polynomial having three roots 1,2,3.
By Factor Theorem, $g(x)=(x-1)(x-2)(x-3)$.
So $f(x)=(x-1)(x-2)(x-3)+x^2$.
Thus, $f(4)=(4-1)(4-2)(4-3)+4^2=22$.

Answer: C)

18. Which of the following is equal to $\sqrt[3]{9-4\sqrt{5}}+\sqrt[3]{9+4\sqrt{5}}$?
A) $2\sqrt[3]{3}$ B) $2\sqrt{5}$ C) 4 D) 3 E) NOTA

Solution: If we let $A=\sqrt[3]{9-4\sqrt{5}}$ and $B=\sqrt[3]{9+4\sqrt{5}}$, then $A^3+B^3=18$ and $AB=1$. Then $(A+B)^3-3AB(A+B)=18$. The equation can be written as $X^3-3x-18=0$, where $X=\sqrt[3]{9-4\sqrt{5}}+\sqrt[3]{9+4\sqrt{5}}$. Factoring the polynomial, $X^3-3x-18=(X-3)(X^2+3X+6)=0$, we obtain the value of $X=3$.

Answer: D)

19. Find the sum of the solutions to the equation $2^{\sin^2 x}+5\cdot 2^{\cos^2 x}=7$ where x is in the interval $(0,2\pi)$.
A) $\frac{\pi}{2}$ B) π C) $\frac{3\pi}{2}$ D) 2π E) NOTA

Solution: If we let $X=2^{\sin^2 x}$, then the equation becomes $X+5\cdot\frac{2}{X}=7$. Solving the equation for X, we have $X=2$ or $X=5$. Since $0 \leq \sin^2 x \leq 1$, $X \leq 2$. Thus, $X=2$ and equivalently, $\sin^2 x=1$. Solutions are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Answer: D)

20. What is the remainder when $1! + 2! + 3! + \cdots + 2018!$ is divided by 7?
A) 4 B) 5 C) 6 D) 0 E) NOTA

Solution: Note $n!$ Is divisible by 7 for $n \geq 7$. So the remainder is equal to the remainder in the division $1! + 2! + \cdots + 6! \equiv 1 + 2 + (-1) + 3 + 1 + (-1) \equiv 5 \pmod 7$.

Answer: B)

21. If a_n is an increasing arithmetic sequence satisfying $a_5 + a_9 = 0$ and $|a_6| = |a_7| + 2$, what is a_1?
A) -12 B) -10 C) -8 D) -6 E) NOTA

Solution: Since a_n is an arithmetic sequence, $2a_7 = a_5 + a_9 = 0$, $a_7 = 0$. $|a_6| = |a_7| + 2 = 0 + 2 = 2$. Since $a_6 < a_7$, $a_6 = -2$. Hence $a_n = -12 + (n-1)(-2)$.

Answer: A)

22. If x is a positive real number such that $\sin(\arctan(\frac{x}{2})) = \frac{x}{3}$, what is the value of x?
A) 1 B) 2 C) $\sqrt{5}$ D) $\sqrt{6}$ E) NOTA

Solution: If we let $\theta = \arctan\left(\frac{x}{2}\right)$, then $\tan\theta = \frac{x}{2}$ where θ is an angle in the first quadrant. Then $\sin\theta = \sin\left(\arctan\left(\frac{x}{2}\right)\right) = \frac{x}{\sqrt{x^2+4}}$. From the given condition we obtain $\frac{x}{3} = \frac{x}{\sqrt{x^2+4}}$. Solving it for x, we have $x = \sqrt{5}$.

Answer: C)

23. Let a_n be a sequence of integers. Suppose that a_1, a_2, a_3 form an arithmetic sequence and a_2, a_3, a_4 form a geometric sequence with an integer common ratio. If $a_4 - a_1 = 30$, what is $a_1 + a_2 + a_3 + a_4$?
A) 24 B) 33 C) 36 D) 46 E) NOTA

Solution: Let $a_2 = a$ and $a_3 = ar$ where r is a common ratio. Then we can write $a_1 = 2a - ar$ and $a_4 = ar^2$. Then $a_4 - a_1 = ar^2 - (2a - ar) = a(r^2 + r - 2) = a(r+2)(r-1) = 30$. Since both $r+2$ and $r-1$ are divisors of 30 and differ by 3, $r - 1 = 2$ and $r + 2 = 5$. So $r = 3$ and $a = 3$. Thus the four terms are $-3,3,9,27$. So their sum is $-3 + 3 + 9 + 27 = 36$.

Answer: C)

24. Assume that the system of equations $\begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} x \\ y \end{bmatrix}$ has a solution $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x^2 + y^2 = 1$. What is the sum of all possible values of k?
A) 1 B) 3 C) 5 D) 10 E) NOTA

Solution: In order for the system to have nontrivial solutions, the matrix $\begin{bmatrix} 2-k & 6 \\ 2 & 1-k \end{bmatrix}$ has to be nonsingular. In other words, $\det\left(\begin{bmatrix} 2-k & 6 \\ 2 & 1-k \end{bmatrix}\right) = \begin{vmatrix} 2-k & 6 \\ 2 & 1-k \end{vmatrix} = (2-k)(1-k) - 12 = 0$. Solving the equation for k, we have $k = 5, -2$.

Answer: B)

25. Find the remainder when $3^{21} + 7^{21}$ is divided by 25.
A) 0 B) 3 C) 8 D) 10 E) NOTA

Solution: If $x = 5$, then $3^{21} + 7^{21} = (x-2)^{21} + (x+2)^{21}$.
By Binomial Theorem, $(x-2)^{21} + (x+2)^{21} = 2x^{21} + 2\binom{21}{19}x^{19} + \cdots + 2\binom{21}{3}x^3 + 2\binom{21}{1}x$. From this we know that the remainder is equal to the remainder dividing $2\binom{21}{1}x = 2\binom{21}{1}5 = 210$ by 25, which is equal to 10.

Answer: D)

26. If $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, what is the value of $(1-z)(1-z^2)(1-z^3)(1-z^4)$?
A) 2 B) 3 C) 4 D) 5 E) NOTA

Solution: Note that z, z^2, z^3, z^4 are four roots of $x^4 + x^3 + x^2 + x + 1 = 0$. By Factor Theorem, $x^4 + x^3 + x^2 + x + 1 = (x-z)(x-z^2)(x-z^3)(x-z^4)$. Substituting $x = 1$, we obtain $(1-z)(1-z^2)(1-z^3)(1-z^4) = 5$.

Answer: D)

27. When $\sqrt{15 \cdot 17 \cdot 19 \cdot 21 + 16}$ is simplified, it is a three-digit integer. What is the sum of the digits?
A) 9 B) 12 C) 15 D) 18 E) NOTA

Solution: If $x = 18$, then $\sqrt{15 \cdot 17 \cdot 19 \cdot 21 + 16}$

$$= \sqrt{(x-3)\cdot(x-1)\cdot(x+1)\cdot(x+3) + 16} =$$
$$= \sqrt{(x^2-9)(x^2-1) + 16} = \sqrt{x^4 - 10x^2 + 25}$$
$$= x^2 - 5 = 18^2 - 5 = 319$$

Answer: E)

28. What is the largest integer less than or equal to the sum $\sum_{n=1}^{2018} \log_2\left(1 + \frac{1}{n}\right)$?
A) 10 B) 11 C) 12 D) 13 E) NOTA

Solution: $\log_2\left(1+\frac{1}{1}\right)+\log_2\left(1+\frac{1}{2}\right)+\log_2\left(1+\frac{1}{3}\right)+\cdots+\log_2(1+\frac{1}{2018})=$
$\log_2 2 + \log_2\frac{3}{2}+\log_2\frac{4}{3}+\cdots+\log_2\frac{2019}{2018}=\log_2 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{2019}{2018}=\log_2 2019=10.\cdots$.

Answer: A)

29. If a, b, c are positive real numbers such that $a^3+b^3+c^3=3abc$, what is the value of $\frac{(a+b)(b+c)(c+a)}{abc}$?
A) 8 B) 4 C) 2 D) 1 E) NOTA

Solution: Note that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. So $a^3+b^3+c^3=3abc$ if and if $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$. Since a, b, c are positive real numbers, $a^2+b^2+c^2-ab-bc-ca=0$ which implies $a=b=c$. Therefore, $\frac{(a+b)(b+c)(c+a)}{abc}=\frac{2a\cdot 2b\cdot 2c}{abc}=8$.

Answer: A)

30. If a nonzero 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $A^2=A$ and $A\neq\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which of following statements about A is NOT true?
A) $ad-bc=0$
B) $a+d=1$
C) $A^{2018}=A$
D) $A^T=A$
E) NOTA

Solution: If $A^2=A$, then A is singular, so $\det(A)=ad-bc=0$. The trace of A , $tr(A)=a+d$, is equal to 1. $A^n=A$ for each positive integer n. But A is not necessarily symmetric.

Answer: D)